MATH2040 Linear Algebra II

Tutorial 3

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1 Examples:

Example 1

For $\mathbb{F} = \mathbb{R}$, let $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$.

(a) Find the eigenvalues and eigenvectors of A.

(b) Determine whether A is diagonalizable, and if yes, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$

Solution

(a)

$$\begin{aligned} f(\lambda) = \det(A - \lambda I) &= \det \begin{pmatrix} -\lambda & -2 & -3 \\ -1 & 1-\lambda & -1 \\ 2 & 2 & 5-\lambda \end{pmatrix} \\ &= \det \begin{pmatrix} -\lambda & -2 & -3 \\ -1 & 1-\lambda & -1 \\ 2-\lambda & 0 & 2-\lambda \end{pmatrix} \\ &= 2\det \begin{pmatrix} -1 & -1 \\ 2-\lambda & 2-\lambda \end{pmatrix} + (1-\lambda)\det \begin{pmatrix} -\lambda & -3 \\ 2-\lambda & 2-\lambda \end{pmatrix} \\ &= (1-\lambda)(2-\lambda)(3-\lambda). \end{aligned}$$

So the eigenvalues are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$.

$$\begin{aligned} E_{\lambda_1} &= N \begin{pmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{pmatrix} = \operatorname{span} \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\} \\ E_{\lambda_2} &= N \begin{pmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{pmatrix} = \operatorname{span} \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\} \\ E_{\lambda_3} &= N \begin{pmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{pmatrix} = \operatorname{span} \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}. \end{aligned}$$

So the eigenvectors are $v_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

(b) Since the eigenvalues are all distinct, so A is diagonalizable with $Q = (v_1, v_2, v_3) = \begin{pmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Example 2

For each of the following linear operator $T : V \to V$, test T for diagonalizability, and if T is diagonalizable, find an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

(a) $V = \mathbb{R}^3$ and $T(a, b, c) = (6a + 3b - 8c, -2b, a - 3c)$.

(b) $V = P_3(\mathbb{R})$ and for any $p(x) \in P_3(\mathbb{R}), T(p(x)) \in P_3(\mathbb{R})$ is defined by $T(p(x)) = p(x) + p(2)x$.

Solution

(a) Let α be the standard ordered basis, then $[T]_\alpha = \begin{pmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix}$, and we need to find the eigenvalues and eigenspaces of $[T]_\alpha$.

Note

$$f(\lambda) = \det \begin{pmatrix} 6-\lambda & 3 & -8 \\ 0 & -2-\lambda & 0 \\ 1 & 0 & -3-\lambda \end{pmatrix} = (-2-\lambda)\det \begin{pmatrix} 6-\lambda & -8 \\ 1 & -3-\lambda \end{pmatrix} = -(2+\lambda)^2(\lambda - 5).$$

So the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 5$. Note λ_1 is a root with multiplicity 2.

Then we consider

$$N([T]_\alpha - \lambda_1 I) = N \begin{pmatrix} 8 & 3 & -8 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$
$$N([T]_\alpha - \lambda_2 I) = N \begin{pmatrix} 1 & 3 & -8 \\ 0 & -7 & 0 \\ 1 & 0 & -8 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 8 \\ 0 \\ 1 \end{pmatrix} \right\},$$

so corresponding eigenspaces are

$$E_{\lambda_1} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$
$$E_{\lambda_2} = \text{span}\left\{ \begin{pmatrix} 8 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Since $\dim(E_{\lambda_1}) < m_1 = 2$, so T is not diagonalizable.

(b) Let α be the standard ordered basis, then $[T]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 4 & 8 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, and we need to find the eigenvalues and eigenspaces of $[T]_\alpha$.

Note

$$f(\lambda) = \det \begin{pmatrix} 1-\lambda & 0 & 0 & 0 \\ 1 & 3-\lambda & 4 & 8 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^3(3-\lambda).$$

So the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$. Note λ_1 is a root with multiplicity 3.

Then we consider

$$N([T]_\alpha - \lambda_1 I) = N\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -8 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$N([T]_\alpha - \lambda_2 I) = N\begin{pmatrix} -2 & 0 & 0 & 0 \\ 1 & 0 & 4 & 8 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

so corresponding eigenspaces are

$$E_{\lambda_1} = \text{span}\left\{x - 2, x^2 - 4, x^3 - 8\right\}$$
$$E_{\lambda_2} = \text{span}\left\{x\right\}$$

Since $\dim(E_{\lambda_1}) = m_1 = 3$, so T is diagonalizable. Therefore, $\beta = \{x - 2, x^2 - 4, x^3 - 8, x\}$.

2 Exercises:

Question 1 (Section 5.2 Q2(e)):

For $\mathbb{F} = \mathbb{R}$, let $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$. Determine whether A is diagonalizable, and if yes, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

Question 2 (Section 5.2 Q3(b)):

Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $T(ax^2+bx+c) = cx^2+bx+a$. Test T for diagonalizability, and if T is diagonalizable, find an ordered basis β for $P_2(\mathbb{R})$ such that $[T]_\beta$ is a diagonal matrix.

Question 3 (Section 5.2 Q12):

Let T be an invertible linear operator on a finite-dimensional vector space V.

(a) Prove that the eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}.

(b) Prove that if T is diagonalizable, then T^{-1} is diagonalizable.

Solution

(Please refer to the practice problem set 3.)